1. Estimators

Statistical inference is the process of using sample results to draw conclusions about the characteristics of a population.

A point estimate is a single value derived from sample data that estimates the value of some population parameter. Two important attributes of a good point estimate are consistency and absence of bias.

- **Bias.** A sample statistic is an unbiased estimator of some population parameter if the expected value of the statistic is equal to the population parameter. That is, if we were to consider many samples of the same size drawn from the same population, the average value of the sample statistic would converge on the true value of the population parameter.

- **Consistency.** A statistic is consistent if, as the sample size increases, the variation of the sample statistic (such as sample mean) consistently decreases, so that the sample statistic becomes an increasingly better estimator of the population parameter.

2. Credible Intervals and Confidence Intervals

We said above that the purpose of statistical inference is to draw conclusions about a population parameter based on information in a sample. Because sample data only gives us limited information about a population, the inferences we draw from a given sample about a population parameter are only uncertain. One way to express this uncertainty is as a range (or probability distribution) of likely values. That is the purpose of credible intervals and confidence intervals.

Definition: credible intervals and confidence intervals are two ways of expressing a statistical estimate in terms of a range of likely values.

**Credible Intervals vs. Confidence Intervals**

For the last 75 years or so, people have been taught to use confidence intervals. However, credible intervals are arguably a better way to look at things. We can see this by comparing their formal definitions. For example, suppose we want to know the population mean of some variable, and estimate this from sample data. What can we infer with, say, 95% certainty:

- The 95% credible interval is the range of values in which we are 95% certain the population mean, \( \mu \), falls, based on our sample data of size \( n \).
The 95% confidence interval is the range of values such that, if all possible samples of the same size \( n \) are taken, 95% of them include the true population mean somewhere within the interval around their sample means, and only 5% of them do not.

The points to appreciate here are: (1) that the credible interval tells us exactly what we are trying to find out – i.e., it is perfectly aligned with the logic of statistical inference, viz. to draw inferences about a population parameter from sample data; and (2) the formal definition of the confidence interval is horribly convoluted and has little to do with common-sense notions about why we’re collecting sample data in the first place.

The logical difference between the two approaches is illustrated by the diagram above.

- Credible intervals tell us what we actually want to know: the likely value of a population parameter (e.g., \( \Theta \)), given a sample statistic (e.g., \( X \)). That is, a credible interval answers the question: what is \( P(\Theta | X) \) – or, say, in the case of inferences about a population mean, what is \( P(\mu | \bar{X}) \).

- Confidence intervals address the reverse question: what is the likely value of a sample statistic (\( X \)) given some population parameter (\( \Theta \)). That is, confidence intervals address the question: what is \( P(X | \Theta) \); or in the case of means \( P(\bar{X} | \mu) \).

The issue here is why we would ever want to do the latter? If we know a population parameter, why would we be interested in estimating a sample statistic? For one thing, this reverses the basic logic of statistical inference, which is to estimate unknown, population parameters. Second, we already know the sample statistic: we can calculate it from sample data.

The real reason people continue to use confidence intervals, ironically, is that, under many conditions, they are identical to credible intervals. That is, people for decades have simply constructed confidence intervals, and then interpreted them as credible intervals.

In this lecture and the next concerning inferences about a population mean, credible intervals and confidence intervals are identical (under some mild assumptions, which we will not
discuss). Therefore, although in Chapter 8 the text and formulas are for "confidence intervals", we can understand these to also apply to credible intervals.

3. Credible Intervals and Confidence Intervals of the Population Mean (σ Known)

Problem: we have captured, measured, weighed, and released 100 sea otter pups in Monterey Bay. We wish to estimate the mean weight for the entire population of sea otter pups in Monterey Bay. We also know (from some other source) the population standard deviation.

Step 1. Let \( p \) be the width of our credible interval (expressed as a proportion). This means we want a credible interval in which we believe with \((p \times 100)\%\) certainty that the true population mean falls. For example, with \( p = .95 \), we will produce a \(.95 \times 100 = 95\%\) credible interval for the population mean.

Step 2. Calculate the areas in the upper and lower tail of a standard normal distribution corresponding to proportion \( p \).

\[
\alpha = 1 - p \\
\alpha / 2 = \text{area in upper tail} = \text{area in lower tail}
\]

For example, if \( p = .95 \), \( \alpha = .05 \), \( \alpha / 2 = .025 \)

Step 3. Calculate the \( z \)-values (\( z_L \) and \( z_U \)) that define the upper and lower limits of the credible interval:

Lower limit: \( z_L = z \)-value such that the proportion of the area below is \( \alpha / 2 \)
Upper limit: \( z_U = z \)-value such that proportion of the area above is \( 1 - \alpha / 2 \).
For this, we can use the norm.inv function of Excel. For example:

\[ z_L = \text{norm.inv (.025)} \]
\[ z_U = \text{norm.inv (.975)} \]

So \( z_L = -1.96 \) and \( z_U = 1.96 \).

Step 4. Convert \( z_L \) and \( z_U \) to the scale of your actual data.

Recall the formula for a z-score: \( z = \frac{X - \mu}{\sigma} \) and solve for \( X \). (We will use \( \bar{X} \), the sample mean in place of \( \mu \); and the standard error of the mean, \( \sigma_{\bar{X}} \) instead of \( \sigma \). This is because we are concerned here not with a credible interval of the population values, but of the population mean – therefore we base our intervals on the estimated variability of sample means.

Lower limit of CI = \( \bar{X} + z_L \times \sigma_{\bar{X}} \)
Upper limit of CI = \( \bar{X} + z_U \times \sigma_{\bar{X}} \)

Recall that \( \sigma_{\bar{X}} \) is the standard error of the mean, equal to \( \sigma / \sqrt{n} \), where \( \sigma \) is the population standard deviation and \( n \) is the sample size.

Example. The mean weight in a sample of 25 sea otter pups is 800g. The population standard deviation is 100g. What is the 95% credible interval for the population mean (\( \mu \)) weight?

\[ p = .95 \]
\[ \alpha = 1 - .95 = .05 \]
\[ \alpha/2 = .025 \]

\[ z_L = \text{norm.inv (.025)} = -1.96 \]
\[ z_U = \text{norm.inv (.975)} = 1.96 \]

\[ \sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{100}{\sqrt{25}} = 20 \]

Lower limit of CI = \( \bar{X} + z_L \times \sigma_{\bar{X}} = 800 + (-1.96 \times 20) = 760.8g \)
Upper limit of CI = $\bar{x} + Z_u \times \sigma_\bar{x} = 800 + (1.96 \times 20) = 839.2g$

Watch Video:
http://www.youtube.com/watch?v=bekNKJoxYbQ

Entire series (Google 'Khan academy confidence intervals'):
https://www.khanacademy.org/math/probability/statistics-inferential/confidence-intervals/v/confidence-interval-1

**Homework**

Read: pp. 365–370

Problem: For a sample of size $n = 64$, the sample mean $\bar{x} = 85$. The population standard deviation, $\sigma = 8$. Set up a 99% credible interval for the population mean, $\mu$. 